Inverse Problem for the Schrödinger Operator in an Unbounded Strip

Laure Cardoulis, Michel Cristofol, Patricia Gaitan

Inverse Problem for the Schrödinger Operator in an Unbounded Strip

Laure Cardoulis* Michel Cristofol† Patricia Gaitan‡

Abstract

We consider the operator $H := i\partial_t + \nabla \cdot (c\nabla)$ in an unbounded strip Ω in $\mathbb{R}^2$, where $c(x,y) \in \mathcal{C}^3(\overline{\Omega})$. We prove an adapted global Carleman estimate and an energy estimate for this operator. Using these estimates, we give a stability result for the diffusion coefficient $c(x,y)$.

AMS 2000 subject classification: 35J10, 35R30.

1 Introduction

Let $\Omega = \mathbb{R} \times (-\frac{d}{2}, \frac{d}{2})$ be an unbounded strip of $\mathbb{R}^2$ with a fixed width d. We will consider the Schrödinger equation

$$(1.1) \qquad \begin{cases} Hq := i\partial_t q + \nabla \cdot (c(x,y)\nabla q) = 0 \text{ in } Q = \Omega \times (0,T), \\ q(x,y,t) = b(x,y,t) \text{ on } \Sigma = \partial\Omega \times (0,T), \\ q(x,y,0) = q_0(x,y) \text{ on } \Omega, \end{cases}$$

where $c(x,y) \in \mathcal{C}^3(\overline{\Omega})$ and $c(x,y) \geq c_{min} > 0$. Moreover, we assume that c and all its derivatives up to order three are bounded. If we assume that q_0 belongs to $H^4(\Omega)$ and b is sufficiently regular (e.g. $b \in H^1(0,T,H^{\frac{9}{2}+\varepsilon}(\partial\Omega)) \cap H^2(0,T,H^{\frac{5}{2}+\varepsilon}(\partial\Omega))$ and some additional conditions), then (1.1) admits a solution in $H^1(0,T,H^{\frac{3}{2}+\varepsilon}(\Omega))$. We will use this regularity result later. The aim of this paper is to give a stability and uniqueness result for the coefficient $c(x,y)$ using global Carleman estimates and energy estimates. We denote by ν the outward unit normal to Ω on $\Gamma = \partial\Omega$. We denote $\Gamma = \Gamma^+ \cup \Gamma^-$, where $\Gamma^+ = \{(x,y) \in \Gamma;\ y = \frac{d}{2}\}$ and $\Gamma^- = \{(x,y) \in \Gamma;\ y = -\frac{d}{2}\}$. We use the following notations $\nabla \cdot (c\nabla u) = \partial_x(c\partial_x u) + \partial_y(c\partial_y u)$, $\nabla u \cdot \nabla v = \partial_x u \partial_x v + \partial_y u \partial_y v$, $\partial_\nu u = \nabla u \cdot \nu$.
We shall use the following notations $Q = \Omega \times (0,T)$, $\widetilde{Q} = \Omega \times (-T,T)$, $\Sigma = \Gamma \times (0,T)$, $\widetilde{\Sigma} = \Gamma \times (-T,T)$, $\Lambda(R_1) := \{\Phi \in L^\infty(\Omega), 0 < R_1 \leq \|\Phi\|_{L^\infty(\Omega)}\}$, and $\Lambda(R_2) := \{\Phi \in L^\infty(\Omega), \|\Phi\|_{L^\infty(\Omega)} \leq R_2\}$, where R_1 and R_2 are positive constants with $R_1 \leq R_2$.

Our problem can be stated as follows:
Is it possible to determine the coefficient $c(x,y)$ from the measurement of $\partial_\nu(\partial_t q)$ on Γ^+?

Let q (resp. $\widetilde{q}$) be a solution of (1.1) associated with (c, b, q_0) (resp. ($\widetilde{c}$, b, q_0)) satisfying some regularity properties:

- $\partial_t\widetilde{q}$, $\nabla(\partial_t\widetilde{q})$ and $\Delta(\partial_t\widetilde{q})$ are in $\Lambda(R_2)$,

*Université de Toulouse 1, UMR 5640, Ceremath/MIP, Place Anatole France, 31000 Toulouse, France, laure.cardoulis@univ-tlse1.fr
†Université de Provence, CMI,UMR CNRS 6632, 39, rue Joliot Curie,13453 Marseille Cedex 13 France, Université Paul Cézanne, IUT de Marseille, France, cristo@cmi.univ-mrs.fr
‡Université de Provence, CMI,UMR CNRS 6632, 39, rue Joliot Curie,13453 Marseille Cedex 13, France, Université de la Méditerranée, IUT d'Aix en Provence, France, gaitan@cmi.univ-mrs.fr

- q_0 is a real valued function in $\mathcal{C}^3(\Omega)$,
- q_0 and all its derivatives up to order three are in $\Lambda(R_2)$.

Our main result is

$$|c-\widetilde{c}|^2_{H^1(\Omega)} \leq C|\partial_\nu(\partial_t q) - \partial_\nu(\partial_t \widetilde{q})|^2_{L^2((0,T)\times\Gamma^+)},$$

where C is a positive constant which depends on $(\Omega, \Gamma, T, R_1, R_2)$ and where the above norms are weighted Sobolev norms.
The major novelty of this paper is to give an H^1 stability estimate for the diffusion coefficient with only one observation in an unbounded domain.
We prove an adapted global Carleman estimate and an energy estimate for the operator H with a boundary term on Γ^+. Such energy estimate has been proved in [23] for the Schrödinger operator in a bounded domain in order to obtain a controllability result. Then using these estimates and following the method developed by Imanuvilov, Isakov and Yamamoto for the Lamé system in [16], [17], we give a stability and uniqueness result for the diffusion coefficient $c(x, y)$. Note that this stability result corresponds to a stability result for three linked coefficients (c, $\partial_y c$ and $\partial_y c$) with only one observation. For independent coefficients, in our knowledge, there is no stability result with one observation.
The method of Carleman estimates was introduced in the field of inverse problems in the works of Bukhgeim and Klibanov (see [1], [3], [19], [20]). The first stability result for a multidimensional inverse problem (for a hyperbolic equation) was obtained by Puel and Yamamoto [24] using a modification of the idea of [3].
For the non stationnary Schrödinger equation, [2] gives a stability result for the potential in a bounded domain. For the stationnary Schrödinger equation, we can cite recent results concerning uniqueness for the potential from partial Cauchy data (see for exemple [18] and the references herein).
In unbounded domains Carleman estimate with an internal observation has been proved for the heat equation in [6].
A physical background could be the characterization of the diffusion coefficient for a strip in geophysics. Indeed if we look for time harmonic solutions of (1.1), the problem can be written, after some changes of variables as the reconstruction of a non local potential P in a strip for the operator $-\Delta + P$. Few results for inverse problems exist in a two-dimensional strip (see [9]). For the layer $\mathbb{R}^n \times [0, h]$ with $n \geq 2$, several results exist for the stationnary inverse problems (see [5], [10], [8], [13], [15], [25], ...).
On the other hand, we can link our problem to the determination of the curvature function for a curved quantum guide (see [12], [7], [11], ...).
This paper is organized as follows. In section 2, we give an adapted global Carleman estimate for the operator H. In section 3, we prove an energy estimate and we give a stability result for the diffusion coefficient c.

2 Global Carleman Estimate

Let $c = c(x, y)$ be a bounded positive function in $\mathcal{C}^3(\overline{\Omega})$ such that

Assumption 2.1. *$c(x, y) \in \Lambda(R_1)$, c and all its derivatives up to order three are in $\Lambda(R_2)$.*

Let $q = q(x, y, t)$ be a function equals to zero on $\partial\Omega \times (-T, T)$ and solution of the Schrödinger equation

$$i\partial_t q + \nabla \cdot (c(x, y)\nabla q) = f.$$

We prove here a global Carleman-type estimate for q with a single observation acting on a part Γ^+ of the boundary Γ in the right-hand side of the estimate. Let $\widetilde{\beta}$ be a $\mathcal{C}^4(\overline{\Omega})$ positive function such that there exists positive constant C_{pc} which satisfies

Assumption 2.2.
- *$|\nabla\widetilde{\beta}| \in \Lambda(R_1)$, $\partial_\nu\widetilde{\beta} \leq 0$ on Γ^-,*
- *$\widetilde{\beta}$ and all its derivatives up to order four are in $\Lambda(R_2)$.*
- *$2\Re(D^2\widetilde{\beta}(\zeta, \bar{\zeta})) - c\nabla c \cdot \nabla\widetilde{\beta}|\zeta|^2 + 2c^2|\nabla\widetilde{\beta} \cdot \zeta|^2 \geq C_{pc}|\zeta|^2$, for all $\zeta \in \mathbb{C}$*

where

$$D^2\widetilde{\beta} = \begin{pmatrix} c\partial_x(c\partial_x\widetilde{\beta}) & c\partial_x(c\partial_y\widetilde{\beta}) \\ c\partial_y(c\partial_x\widetilde{\beta}) & c\partial_y(c\partial_y\widetilde{\beta}) \end{pmatrix}.$$

Note that the last assertion of Assumption 2.2 expresses the pseudo-convexity condition for the function $\widetilde{\beta}$. This Assumption imposes restrictive conditions for the choice of the functions $c(x,y)$ in connection with the function $\widetilde{\beta}$. Note that there exists functions satisfying such Assumptions; indeed, if we consider

$$c(x,y) \in \left\{ f \in C^1(\Omega); \exists r_0 \text{ positive constant}, \left\{ \begin{array}{l} -f\ \partial_y f \partial_y \widetilde{\beta} \geq r_0 > 0, \\ f\ \partial_y f \partial_y \widetilde{\beta}((\frac{\partial_x f}{\partial_y f})^2 + 1) + 2f^2(\partial_{yy}\widetilde{\beta} + (\partial_y \widetilde{\beta})^2) \geq r_0 > 0. \end{array} \right. \right\}$$

then a function $\widetilde{\beta}(x,y) = \widetilde{\beta}(y)$ is available (for example, $c(x,y) = (\frac{1}{1+x^2} + 1)e^{-y}$ and $\widetilde{\beta}(x,y) = e^y$).

Similar restrictive conditions have been highlighted for the hyperbolic case in [21], [22] and for the Schrödinger operator in [14]

Then, we define $\beta = \widetilde{\beta} + K$ with $K = m\|\widetilde{\beta}\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t \in (-T,T)$, we define the following weight functions

$$\varphi(x,y,t) = \frac{e^{\lambda\beta(x,y)}}{(T+t)(T-t)}, \qquad \eta(x,y,t) = \frac{e^{2\lambda K} - e^{\lambda\beta(x,y)}}{(T+t)(T-t)}. \tag{2.1}$$

Let H be the operator defined by

$$Hq := i\partial_t q + \nabla\cdot(c(x,y)\nabla q) \ \text{ in } \ \widetilde{Q} = \Omega \times (-T,T). \tag{2.2}$$

We set $\psi = e^{-s\eta}q$, $M\psi = e^{-s\eta}H(e^{s\eta}\psi)$ for $s > 0$ and we introduce the following operators

$$M_1\psi := i\partial_t\psi + \nabla\cdot(c\nabla\psi) + s^2 c|\nabla\eta|^2\psi, \tag{2.3}$$

$$M_2\psi := is\partial_t\eta\psi + 2cs\nabla\eta\cdot\nabla\psi + s\nabla\cdot(c\nabla\eta)\psi. \tag{2.4}$$

Then the following result holds.

Theorem 2.3. *Let H, M_1, M_2 be the operators defined respectively by (2.2), (2.3), (2.4). We assume that Assumptions 2.1 and 2.2 are satisfied. Then there exist $\lambda_0 > 0$, $s_0 > 0$ and a positive constant $C = C(\Omega,\Gamma,T,C_{pc},R_1,R_2)$ such that, for any $\lambda \geq \lambda_0$ and any $s \geq s_0$, the next inequality holds:*

$$\begin{aligned} & s^3\lambda^4 \int_{-T}^{T}\int_\Omega e^{-2s\eta}\varphi^3|q|^2\ dx\ dy\ dt + s\lambda\int_{-T}^{T}\int_\Omega e^{-2s\eta}\varphi|\nabla q|^2\ dx\ dy\ dt + \|M_1(e^{-s\eta}q)\|^2_{L^2(\widetilde{Q})} \\ & + \|M_2(e^{-s\eta}q)\|^2_{L^2(\widetilde{Q})} \leq C\left[s\lambda\int_{-T}^{T}\int_{\Gamma^+} e^{-2s\eta}\varphi|\partial_\nu q|^2\ \partial_\nu\beta\ d\sigma\ dt + \int_{-T}^{T}\int_\Omega e^{-2s\eta}\ |Hq|^2\ dx\ dy\ dt\right], \end{aligned} \tag{2.5}$$

for all q satisfying $Hq \in L^2(\Omega\times(-T,T))$, $q \in L^2(-T,T;H_0^1(\Omega))$, $\partial_\nu q \in L^2(-T,T;L^2(\Gamma))$.

Proof. If we set $\psi = e^{-s\eta}q$, we calculate $M\psi = e^{-s\eta}H(e^{s\eta}\psi)$ and we obtain:

$$M\psi = M_1\psi + M_2\psi$$

with M_1 and M_2 defined respectively by (2.3) and (2.4). Then

$$\begin{aligned} \int_{-T}^{T}\int_\Omega |M\psi|^2 dx\ dy\ dt &= \int_{-T}^{T}\int_\Omega |M_1\psi|^2 dx\ dy\ dt + \int_{-T}^{T}\int_\Omega |M_2\psi|^2 dx\ dy\ dt \\ &+ 2\Re(\int_{-T}^{T}\int_\Omega M_1\psi\ \overline{M_2\psi}\ dx\ dy\ dt), \end{aligned} \tag{2.6}$$

where $\overline{z}$ is the conjugate of z, $\Re\ (z)$ its real part and $\Im\ (z)$ its imaginary part. We have to compute the scalar product in (2.6)

$$\Re(\int_{-T}^{T}\int_\Omega M_1\psi\ \overline{M_2\psi}\ dx\ dy\ dt) = I_{11'} + I_{12'} + I_{13'} + I_{21'} + I_{22'} + I_{23'} + I_{31'} + I_{32'} + I_{33'}.$$

Then, we have

$$I_{11'} = \Re\left(\int_{-T}^{T}\int_{\Omega}(i\partial_t\psi)(-is\partial_t\eta\ \overline{\psi})\ dx\ dy\ dt\right) = -\frac{s}{2}\int_{-T}^{T}\int_{\Omega}\partial_{tt}\eta\ |\psi|^2\ dx\ dy\ dt. \tag{2.7}$$

$$\begin{aligned} I_{12'} &= \Re\left(2is\int_{-T}^{T}\int_{\Omega}c\ \partial_t\psi\nabla\eta\cdot\nabla\overline{\psi}\ dx\ dy\ dt\right) \\ &= s\Im\left(\int_{-T}^{T}\int_{\Omega}c\ \nabla\eta\cdot\nabla\psi\ \partial_t\overline{\psi}\ dx\ dy\ dt\right) - s\Im\left(\int_{-T}^{T}\int_{\Omega}c\ \nabla\eta\cdot\nabla\overline{\psi}\ \partial_t\psi\ dx\ dy\ dt\right). \end{aligned}$$

After an integration by parts with respect to the space variable in the first integral and to the time variable in the second integral, we obtain

$$I_{12'} = -s\Im\left(\int_{-T}^{T}\int_{\Omega}\nabla\cdot(c\nabla\eta)\ \psi\ \partial_t\overline{\psi}\ dx\ dy\ dt\right) + s\Im\left(\int_{-T}^{T}\int_{\Omega}c\psi\ \nabla(\partial_t\eta)\cdot\nabla\overline{\psi}\ dx\ dy\ dt\right). \tag{2.8}$$

$$I_{13'} = s\Im\left(\int_{-T}^{T}\int_{\Omega}\nabla\cdot(c\nabla\eta)\ \psi\ \partial_t\overline{\psi}\ dx\ dy\ dt\right). \tag{2.9}$$

Note that $I_{13'}$ vanishes with the first term of $I_{12'}$.

$$\begin{aligned} I_{21'} &= \Re\left(-is\int_{-T}^{T}\int_{\Omega}\partial_t\eta\ \overline{\psi}\ \nabla\cdot(c\nabla\psi)dx\ dy\ dt\right) \\ &= s\Im\left(\int_{-T}^{T}\int_{\Omega}c\psi\ \nabla(\partial_t\eta)\cdot\nabla\overline{\psi}\ dx\ dy\ dt\right). \end{aligned} \tag{2.10}$$

Using integrations by parts, we obtain

$$\begin{aligned} I_{22'} &= 2s\Re\left(\int_{-T}^{T}\int_{\Omega}c\nabla\cdot(c\nabla\psi)\nabla\eta\cdot\nabla\overline{\psi})dx\ dy\ dt\right) \\ &= -s\lambda\int_{-T}^{T}\int_{\Omega}\varphi|\nabla\psi|^2(\nabla\cdot(c^2\nabla\beta)+\lambda c^2|\nabla\beta|^2)\ dx\ dy\ dt \\ &+ s\int_{-T}^{T}\int_{\partial\Omega}c^2\partial_\nu\eta\ |\partial_\nu\psi|^2\ d\sigma\ dt + 2s\lambda^2\int_{-T}^{T}\int_{\Omega}\varphi c^2|\nabla\beta\cdot\nabla\psi|^2dx\ dy\ dt \\ &+ 2s\lambda\Re\left(\int_{-T}^{T}\int_{\Omega}\varphi c\sum_{i,j=1}^{2}\partial_{x_i}(c\partial_{x_j}\beta)\ \partial_{x_i}\psi\ \partial_{x_j}\overline{\psi}\ dx\ dy\ dt\right) \\ &= -s\lambda\int_{-T}^{T}\int_{\Omega}\varphi|\nabla\psi|^2(\nabla\cdot(c^2\nabla\beta)+\lambda c^2|\nabla\beta|^2)\ dx\ dy\ dt \\ &- s\lambda\int_{-T}^{T}\int_{\partial\Omega}c^2\varphi\ \partial_\nu\beta\ |\partial_\nu\psi|^2\ d\sigma\ dt + 2s\lambda^2\int_{-T}^{T}\int_{\Omega}\varphi c^2|\nabla\beta\cdot\nabla\psi|^2dx\ dy\ dt \\ &+ 2s\lambda\Re\left(\int_{-T}^{T}\int_{\Omega}\varphi D^2\beta(\nabla\psi,\nabla\overline{\psi})\ dx\ dy\ dt\right). \end{aligned} \tag{2.11}$$

Using integration by parts, we obtain

$$
\begin{aligned}
(2.12)\quad I_{23'} &= s\Re\left(\int_{-T}^{T}\int_{\Omega}\nabla\cdot(c\nabla\psi)\ \nabla\cdot(c\nabla\eta)\ \overline{\psi}\ dx\ dy\ dt\right)\\
&= s\lambda\int_{-T}^{T}\int_{\Omega}c\varphi\ |\nabla\psi|^2(\nabla\cdot(c\nabla\beta)+\lambda c|\nabla\beta|^2)\ dx\ dy\ dt\\
&- \frac{s\lambda^2}{2}\int_{-T}^{T}\int_{\Omega}\varphi\nabla\cdot(c\nabla\cdot(c\nabla\beta)\nabla\beta))|\psi|^2dx\ dy\ dt\\
&- \frac{s\lambda^3}{2}\int_{-T}^{T}\int_{\Omega}\varphi c|\nabla\beta|^2\nabla\cdot(c\nabla\beta)|\psi|^2dx\ dy\ dt\\
&- \frac{s\lambda^4}{2}\int_{-T}^{T}\int_{\Omega}\varphi c^2|\nabla\beta|^4|\psi|^2dx\ dy\ dt-\frac{s\lambda^3}{2}\int_{-T}^{T}\int_{\Omega}\varphi\nabla\cdot(c^2|\nabla\beta|^2\nabla\beta)|\psi|^2dx\ dy\ dt\\
&- \frac{s\lambda^2}{2}\int_{-T}^{T}\int_{\Omega}\varphi c\nabla\beta\cdot\nabla(\nabla\cdot(c\nabla\beta)+\lambda c|\nabla\beta|^2)|\psi|^2dx\ dy\ dt\\
&- \frac{s\lambda}{2}\int_{-T}^{T}\int_{\Omega}\varphi\nabla\cdot(c\nabla(\nabla\cdot(c\nabla\beta)+\lambda c|\nabla\beta|^2))|\psi|^2dx\ dy\ dt.
\end{aligned}
$$

And we obviously have

$$
(2.13)\qquad I_{31'}=s^3\Re\left(\int_{-T}^{T}\int_{\Omega}c\ (-i\partial_t\eta\ \overline{\psi})|\nabla\eta|^2\psi\ dx\ dy\ dt\right)=0.
$$

$$
\begin{aligned}
(2.14)\qquad I_{32'} &= 2s^3\Re\left(\int_{-T}^{T}\int_{\Omega}c^2|\nabla\eta|^2\psi\ \nabla\eta\cdot\nabla\overline{\psi}\ dx\ dy\ dt\right)\\
&= s^3\int_{-T}^{T}\int_{\Omega}c^2|\nabla\eta|^2\nabla\eta\cdot\nabla|\psi|^2\ dx\ dy\ dt\\
&= s^3\lambda^3\int_{-T}^{T}\int_{\Omega}c\varphi^3\left(|\nabla\beta|^2\nabla\cdot(c\nabla\beta)+\nabla\beta\cdot\nabla(c|\nabla\beta|^2)\right)|\psi|^2dx\ dy\ dt\\
&+ 3s^3\lambda^4\int_{-T}^{T}\int_{\Omega}c^2\varphi^3|\nabla\beta|^4|\psi|^2dx\ dy\ dt.
\end{aligned}
$$

$$
\begin{aligned}
(2.15)\qquad I_{33'} &= s^3\Re\left(\int_{-T}^{T}\int_{\Omega}c\ |\nabla\eta|^2\psi\nabla\cdot(c\nabla\eta)\overline{\psi}\ dx\ dy\ dt\right)\\
&= -s^3\lambda^3\int_{-T}^{T}\int_{\Omega}c\ \varphi^3|\nabla\beta|^2\nabla\cdot(c\nabla\beta)|\psi|^2dx\ dy\ dt\\
&- s^3\lambda^4\int_{-T}^{T}\int_{\Omega}c^2\varphi^3|\nabla\beta|^4|\psi|^2dx\ dy\ dt.
\end{aligned}
$$

So, by (2.7)-(2.15), we get :

$$
\begin{aligned}
\Re\left(\int_{-T}^{T}\int_{\Omega}M_1\psi\ \overline{M_2\psi}\ dx\ dy\ dt\right) &= 2s\lambda^2\int_{-T}^{T}\int_{\Omega}\varphi c^2|\nabla\beta\cdot\nabla\psi|^2dx\ dy\ dt\\
&- s\lambda\int_{-T}^{T}\int_{\Omega}\varphi\ c\ \nabla c\cdot\nabla\beta|\nabla\psi|^2dx\ dy\ dt\\
&+ 2s\lambda\Re\left(\int_{-T}^{T}\int_{\Omega}\varphi D^2\beta(\nabla\psi,\nabla\overline{\psi})\ dx\ dy\ dt\right)\\
&+ 2s^3\lambda^4\int_{-T}^{T}\int_{\Omega}c^2\varphi^3|\nabla\beta|^4|\psi|^2dx\ dy\ dt\\
&- s\lambda\int_{-T}^{T}\int_{\partial\Omega}c^2\varphi\partial_\nu\beta|\partial_\nu\psi|^2\ d\sigma\ dt+X,
\end{aligned}
$$

where

$$
\begin{aligned}
X &= \frac{-s}{2}\int_{-T}^{T}\int_{\Omega}\partial_{tt}\eta\ |\psi|^2\ dx\ dy\ dt + 2s\Im\int_{-T}^{T}\int_{\Omega}c\psi\nabla(\partial_t\eta)\cdot\nabla\overline{\psi}dx\ dy\ dt \\
&- \frac{s\lambda^2}{2}\int_{-T}^{T}\int_{\Omega}\varphi\nabla\cdot(c\nabla\cdot(c\nabla\beta)\nabla\beta))|\psi|^2dx\ dy\ dt \\
&- \frac{s\lambda^3}{2}\int_{-T}^{T}\int_{\Omega}\varphi c|\nabla\beta|^2\nabla\cdot(c\nabla\beta)|\psi|^2dx\ dy\ dt \\
&- \frac{s\lambda^4}{2}\int_{-T}^{T}\int_{\Omega}\varphi c^2|\nabla\beta|^4|\psi|^2dx\ dy\ dt - \frac{s\lambda^3}{2}\int_{-T}^{T}\int_{\Omega}\varphi\nabla\cdot(c^2|\nabla\beta|^2\nabla\beta)|\psi|^2dx\ dy\ dt \\
&- \frac{s\lambda^2}{2}\int_{-T}^{T}\int_{\Omega}\varphi c\nabla\beta\cdot\nabla(\nabla\cdot(c\nabla\beta)+\lambda c|\nabla\beta|^2)|\psi|^2dx\ dy\ dt \\
&- \frac{s\lambda}{2}\int_{-T}^{T}\int_{\Omega}\varphi\nabla\cdot(c\nabla(\nabla\cdot(c\nabla\beta)+\lambda c|\nabla\beta|^2))|\psi|^2dx\ dy\ dt \\
&+ s^3\lambda^3\int_{-T}^{T}\int_{\Omega}\varphi^3c\nabla\beta\cdot\nabla(c|\nabla\beta|^2)|\psi|^2dx\ dy\ dt.
\end{aligned}
$$

Recall that:

$$\|M\psi\|^2_{L^2(\widetilde{Q})} = \|M_1\psi\|^2_{L^2(\widetilde{Q})} + \|M_2\psi\|^2_{L^2(\widetilde{Q})} + 2\Re(M_1\psi,\overline{M_2\psi}).$$

Then :

$$
\begin{aligned}
(2.16)\qquad & \|M_1\psi\|^2_{L^2(\widetilde{Q})} + \|M_2\psi\|^2_{L^2(\widetilde{Q})} + 4s\lambda^2\int_{-T}^{T}\int_{\Omega}\varphi c^2|\nabla\beta\cdot\nabla\psi|^2dx\ dy\ dt \\
&+4s\lambda\Re\left(\int_{-T}^{T}\int_{\Omega}\varphi D^2\beta(\nabla\psi,\nabla\overline{\psi})dx\ dy\ dt\right) - 2s\lambda\int_{-T}^{T}\int_{\Omega}\varphi\ c\ \nabla c\cdot\nabla\beta|\nabla\psi|^2dx\ dy\ dt \\
&+4s^3\lambda^4\int_{-T}^{T}\int_{\Omega}c^2\varphi^3|\nabla\beta|^4|\psi|^2dx\ dy\ dt - 2s\lambda\int_{-T}^{T}\int_{\partial\Omega}c^2\varphi\partial_\nu\beta|\partial_\nu\psi|^2\ d\sigma\ dt \le \|M\psi\|^2_{L^2(\widetilde{Q})} + 2|X|.
\end{aligned}
$$

Taking into account

- $|\widetilde{\beta}| + |\nabla\widetilde{\beta}| + |\nabla(\nabla\cdot(c\nabla\widetilde{\beta}))| + |\nabla\cdot(\nabla(\nabla\cdot(c\nabla\widetilde{\beta})))| \le C(\Omega,\Gamma,T,R_2)$ in Ω,
- $|\partial_{tt}\eta| \le C(T)\varphi^3,\quad |\partial_t\varphi| \le C(T)\varphi^2,\quad \varphi \le C(T)\varphi^3,\quad \varphi^2 \le C(T)\varphi^3,$
- $$\begin{aligned}|s\Im(\int_{-T}^{T}\int_{\Omega}c\psi\nabla(\partial_t\eta)\cdot\nabla\overline{\psi}dx\ dy\ dt)| &\le C(T)s\lambda\int_{-T}^{T}\int_{\Omega}c\varphi|\nabla\beta\cdot\nabla\psi|^2dx\ dy\ dt \\ &+C(T)s\lambda\int_{-T}^{T}\int_{\Omega}c\varphi^3|\psi|^2dx\ dy\ dt,\end{aligned}$$

where $C(\Omega,\Gamma,T,R_2)$ is a positive constant depending upon Ω,Γ,T,R_2 and $C(T)$ is a positive constant depending upon T. Therefore we obtain the following estimation for X:

$$|X| \le C(\Omega,\Gamma,T,R_2)\left[(s\lambda^4+s^3\lambda^3)\int_{-T}^{T}\int_{\Omega}\varphi^3|\psi|^2dx\ dy\ dt + s\lambda\int_{-T}^{T}\int_{\Omega}\varphi|\nabla\beta\cdot\nabla\psi|^2dx\ dy\ dt\right].$$

The two terms of the previous estimate of $|X|$ are neglectable with respect to

$$s^3\lambda^4\int_{-T}^{T}\int_{\Omega}c^2\varphi^3|\nabla\beta|^4|\psi|^2dx\ dy\ dt \quad\text{or}\quad s\lambda^2\int_{-T}^{T}\int_{\Omega}\varphi c^2|\nabla\beta\cdot\nabla\psi|^2dx\ dy\ dt,$$

for s and λ sufficiently large. Using Assumption 2.2, we have

$$4s\lambda\Re\left(\int_{-T}^{T}\int_{\Omega}\varphi D^2\beta(\nabla\psi,\nabla\overline{\psi})dx\ dy\ dt\right) - 2s\lambda\int_{-T}^{T}\int_{\Omega}\varphi\ c\ \nabla c\cdot\nabla\beta|\nabla\psi|^2dx\ dy\ dt$$

$$+4s\lambda^2\int_{-T}^{T}\int_{\Omega}\varphi c^2|\nabla\beta\cdot\nabla\psi|^2 dx\ dy\ dt \geq C_{pc}s\lambda\int_{-T}^{T}\int_{\Omega}\varphi|\nabla\psi|^2 dx\ dy\ dt,$$

so (2.16) becomes

$$\|M_1\psi\|^2_{L^2(\widetilde{Q})}+\|M_2\psi\|^2_{L^2(\widetilde{Q})}+4s^3\lambda^4\int_{-T}^{T}\int_{\Omega}c^2\varphi^3|\nabla\beta|^4|\psi|^2 dx\ dy\ dt$$

$$+s\lambda\int_{-T}^{T}\int_{\Omega}\varphi|\nabla\psi|^2 dx\ dy\ dt \leq 2s\lambda\int_{-T}^{T}\int_{\partial\Omega}c^2\varphi\partial_\nu\beta|\partial_\nu\psi|^2\ d\sigma\ dt+\|M\psi\|^2_{L^2(\widetilde{Q})}.$$

Recall that $\partial_\nu\beta \leq 0$ on Γ^-, $c(x,y)\in\Lambda(R_1)\cap\Lambda(R_2)$, $|\nabla\beta|\in\Lambda(R_1)$ and $\psi=e^{-s\eta}q$, then the proof is complete.

□

3 Inverse Problem

In this section, we establish a stability inequality and deduce a uniqueness result for the coefficient c. The Carleman estimate (2.5) proved in section 2 will be the key ingredient in the proof of such a stability estimate.

Let q be solution of

$$\text{(3.1)}\qquad \begin{cases} i\partial_t q+\nabla\cdot(c\nabla q)=0 & \text{in} \quad \Omega\times(0,T),\\ q(x,y,t)=b(x,y,t) & \text{on} \quad \partial\Omega\times(0,T),\\ q(x,y,0)=q_0(x,y) & \text{in} \quad \Omega,\end{cases}$$

and $\widetilde{q}$ be solution of

$$\text{(3.2)}\qquad \begin{cases} i\partial_t\widetilde{q}+\nabla\cdot(\widetilde{c}\nabla\widetilde{q})=0 & \text{in} \quad \Omega\times(0,T),\\ \widetilde{q}(x,y,t)=b(x,y,t) & \text{on} \quad \partial\Omega\times(0,T),\\ \widetilde{q}(x,y,0)=q_0(x,y) & \text{in} \quad \Omega,\end{cases}$$

where c and $\widetilde{c}$ both satisfy Assumption 2.1. If we set $u=q-\widetilde{q}$, $v=\partial_t u$ and $\gamma=\widetilde{c}-c$, then u and v satisfy respectively

$$\text{(3.3)}\qquad \begin{cases} i\partial_t u+\nabla\cdot(c\nabla u)=\nabla\cdot(\gamma\nabla\widetilde{q}) & \text{in} \quad \Omega\times(0,T),\\ u(x,y,t)=0 & \text{on} \quad \partial\Omega\times(0,T),\\ u(x,y,0)=0 & \text{in} \quad \Omega,\end{cases}$$

$$\text{(3.4)}\qquad \begin{cases} i\partial_t v+\nabla\cdot(c\nabla v)=\nabla\cdot(\gamma\nabla\partial_t\widetilde{q})=f & \text{in} \quad \Omega\times(0,T),\\ v(x,y,t)=0 & \text{on} \quad \partial\Omega\times(0,T),\\ v(x,y,0)=\frac{1}{i}\nabla\cdot(\gamma\nabla q_0) & \text{in} \quad \Omega.\end{cases}$$

Assumption 3.1. *q_0 is a real valued function in $\mathcal{C}^3(\Omega)$*

We extend the function v on $\Omega\times(-T,T)$ by the formula $v(x,y,t)=-\overline{v}(x,y,-t)$ for every $(x,y,t)\in\Omega\times(-T,0)$. Note that this extension is available if the initial data is a real valued function. For a pure imaginary initial data, the right extension is $v(x,y,t)=\overline{v}(x,y,-t)$. Note that these extensions satisfy the previous Carleman estimate.

3.1 Energy Estimate

We assume throughout this section that $\gamma\in H_0^1(\Omega)$. We introduce

$$\text{(3.5)}\qquad \mathbb{E}(t)=\int_{\Omega}e^{-2s\eta(x,y,t)}|\partial_t u(x,y,t)|^2 dx\ dy+\int_{\Omega}\varphi^{-1}(x,y,t)\ e^{-2s\eta(x,y,t)}|\partial_t\nabla u(x,y,t)|^2 dx\ dy.$$

In this section, we will give an estimation of $\mathbb{E}(0)$.

First Step: We first give an estimation of $\int_\Omega e^{-2s\eta(x,y,0)}|\partial_t u(x,y,0)|^2 dx\ dy$.
We set $\psi = e^{-s\eta}v$. With the operator

$$M_1\psi = i\partial_t\psi + \nabla\cdot(c\nabla\psi) + s^2|\nabla\eta|^2\psi, \tag{3.6}$$

we introduce, following [2],

$$\mathcal{I} = 2\Im\left(\int_{-T}^{0}\int_\Omega M_1\psi\ \overline{\psi}\ dx\ dy\ dt\right).$$

Assumption 3.2. *$\partial_t\widetilde{q}$, $\nabla(\partial_t\widetilde{q})$, $\Delta(\partial_t\widetilde{q})$ are in $\Lambda(R_2)$.*

We have the following estimate

Lemma 3.3. *We assume that Assumption 3.2 is satisfied. Then there exists a positive constant $C = C(\Omega,\Gamma,T,R_1,R_2)$ such that for any $\lambda\geq\lambda_0$ and $s\geq s_0$, we have*

$$\mathcal{I} = \int_\Omega e^{-2s\eta(x,y,0)}|\partial_t u(x,y,0)|^2\ dx\ dy$$

and

$$\begin{aligned}|\mathcal{I}| &\leq Cs^{-3/2}\lambda^{-2}\int_\Omega e^{-2s\eta(x,y,0)}(|\gamma|^2+|\nabla\gamma|^2)\ dx\ dy\\ &+Cs^{-1/2}\lambda^{-1}\int_{-T}^{T}\int_{\Gamma^+} e^{-2s\eta}\ \varphi\ \partial_\nu\beta\ |\partial_\nu v|^2\ d\sigma\ dt.\end{aligned}$$

Proof. In a first step, we calculate $\mathcal{I}$

$$\begin{aligned}\mathcal{I} &= 2\Im\left(\int_{-T}^{0}\int_\Omega \left(i\partial_t\psi\ \overline{\psi} + \nabla\cdot(c\nabla\psi)\ \overline{\psi} + s^2|\nabla\eta|^2\psi\ \overline{\psi}\right)\ dx\ dy\ dt\right)\\ &= 2\Re\left(\int_{-T}^{0}\int_\Omega \partial_t\ \psi\overline{\psi}\ dx\ dy\ dt\right)\\ &= \int_{-T}^{0}\int_\Omega \partial_t|\psi|^2\ dx\ dy\ dt\\ &= \int_\Omega |\psi(x,y,0)|^2\ dx\ dy\\ &= \int_\Omega e^{-2s\eta(x,y,0)}|v(x,y,0)|^2\ dx\ dy.\end{aligned}$$

So, we have

$$\mathcal{I} = \int_\Omega e^{-2s\eta(x,y,0)}|\partial_t u(x,y,0)|^2\ dx\ dy.$$

In a second step, we estimate $\mathcal{I}$. Using Young inequality we can write

$$\begin{aligned}|\mathcal{I}| &\leq 2\left(\int_{-T}^{T}\int_\Omega |M_1\psi|^2\ dx\ dy\ dt\right)^{\frac12}\left(\int_{-T}^{T}\int_\Omega e^{-2s\eta}|v|^2\ dx\ dy\ dt\right)^{\frac12}\\ &\leq C(T)s^{-\frac32}\lambda^{-2}\left(\|M_1\psi\|^2_{L^2(\widetilde{Q})} + s^3\lambda^4\int_{-T}^{T}\int_\Omega e^{-2s\eta}\varphi^3|v|^2\ dx\ dy\ dt\right)\end{aligned}$$

with $C(T)$ a positive constant which depends on T. Then with the Carleman estimate (2.5) proved in section 2 we have

$$|\mathcal{I}| \leq Cs^{-\frac32}\lambda^{-2}\left(s\lambda\int_{-T}^{T}\int_{\Gamma^+} e^{-2s\eta}\varphi\partial_\nu\beta|\partial_n v|^2\ d\sigma\ dt + \int_{-T}^{T}\int_\Omega e^{-2s\eta}|\nabla\cdot(\gamma\nabla\partial_t\widetilde{q})\ dx\ dy\ dt\right),$$

where $C = C(\Omega,\Gamma^+,T,R_1,R_2)$ is a positive constant. Using Assumption 3.2, since

$$e^{-2s\eta(x,y,t)} \leq e^{-2s\eta(x,y,0)}\quad \text{for all } t\in(-T,T),$$

we obtain for s and λ sufficiently large the estimate

$$|\mathcal{I}| \leq C\ s^{-3/2}\lambda^{-2} \int_\Omega e^{-2s\eta(x,y,0)}(|\gamma|^2 + |\nabla\gamma|^2)\ dx\ dy \\ +C\ s^{-1/2}\lambda^{-1} \int_{-T}^{T}\int_{\Gamma^+} e^{-2s\eta}\varphi\ \partial_\nu\beta\ |\partial_\nu v|^2\ d\sigma\ dt. \tag{3.7}$$

where $C = C(\Omega, \Gamma^+, T, R_1, R_2)$ is a positive constant. □

Second Step: We then give an estimate of $\int_\Omega \varphi^{-1}(x,y,0)\ e^{-2s\eta(x,y,0)}|\partial_t\nabla u(x,y,0)|^2 dx\ dy$. We denote

$$E(t) := \int_\Omega c\ \varphi^{-1}(x,y,t)\ e^{-2s\eta(x,y,t)}|\nabla v(x,y,t)|^2 dx\ dy, \tag{3.8}$$

where $\varphi^{-1} = \frac{1}{\varphi}$. We give an estimate for $E(0)$ in Theorem 3.5. In a first step we prove the following lemma :

Lemma 3.4. *Let v be solution of (3.4) in the following class*

$$v \in C([0,T], H^1(\Omega)),\ \partial_\nu v \in L^2(0,T,L^2(\Gamma)).$$

Then the following identity holds true

$$E(\tau) - E(\kappa) = -2\Re\left(\int_\kappa^\tau \int_\Omega e^{-2s\eta} f\ \varphi^{-1}\partial_t\overline{v}\ dx\ dy\ dt\right) \\ +\Re\left(\int_\kappa^\tau \int_\Omega c\ e^{-2s\eta}(-4s\lambda + 2\lambda\varphi^{-1})\partial_t\overline{v}\ \nabla\beta\cdot\nabla v\ dx\ dy\ dt\right) \\ -2s\int_\kappa^\tau \int_\Omega c\ e^{-2s\eta}\varphi^{-1}\partial_t\eta|\nabla v|^2\ dx\ dy\ dt + \int_\kappa^\tau \int_\Omega ce^{-2s\eta}\partial_t(\varphi^{-1})|\nabla v|^2,$$

for $f \in H_0^1(\Omega)$.

Proof. Since v is solution of (3.4) note that $\partial_t\overline{v} = i\overline{f} - i\nabla\cdot(c\nabla\overline{v})$. Therefore, we obtain the two following equalities.

$$\Re\int_\kappa^\tau \int_\Omega e^{-2s\eta} f\ \varphi^{-1}\partial_t\overline{v}\ dx\ dy\ dt = \Re\left(-i\int_\kappa^\tau \int_\Omega e^{-2s\eta} f\ \varphi^{-1}\nabla\cdot(c\ \nabla\overline{v})\ dx\ dy\ dt\right), \tag{3.9}$$

$$\Re\left(\int_\kappa^\tau \int_\Omega c\ e^{-2s\eta}\partial_t\overline{v}\ (-4s\lambda + 2\lambda\varphi^{-1})\nabla\beta\cdot\nabla v\ dx\ dy\ dt\right) \\ = \Re\left(i\int_\kappa^\tau \int_\Omega c\ e^{-2s\eta}\overline{f}\ (-4s\lambda + 2\lambda\varphi^{-1})\nabla\beta\cdot\nabla v\ dx\ dy\ dt\right) \\ -\Re\left(i\int_\kappa^\tau \int_\Omega c\ e^{-2s\eta}(-4s\lambda + 2\lambda\varphi^{-1})\nabla\cdot(c\nabla\overline{v})\nabla\beta\cdot\nabla v\ dx\ dy\ dt\right). \tag{3.10}$$

We multiply the first equation of (3.4) by $e^{-2s\eta}\ \varphi^{-1}\partial_t\overline{v}$ and we integrate on $(\kappa,\tau)\times\Omega$, where $[\kappa,\tau] \subset [0,T]$. So, if we consider the real part of the obtained equality, we have

$$0 = \Re\left(i\int_\kappa^\tau \int_\Omega e^{-2s\eta}\ \varphi^{-1}|\partial_t v|^2\ dx\ dy\ dt\right) = \Re\left(\int_\kappa^\tau \int_\Omega e^{-2s\eta}\ f\ \varphi^{-1}\partial_t\overline{v}\ dx\ dy\ dt\right) \\ -\Re\left(\int_\kappa^\tau \int_\Omega e^{-2s\eta}\ \varphi^{-1}\nabla\cdot(c\nabla v)\partial_t\overline{v}\ dx\ dy\ dt\right).$$

Then by integration by parts, we obtain

$$0 = \Re\left(\int_\kappa^\tau \int_\Omega e^{-2s\eta}\ f\ \varphi^{-1}\partial_t\overline{v}\ dx\ dy\ dt\right) + \Re\left(\int_\kappa^\tau \int_\Omega c\nabla v\cdot\nabla(e^{-2s\eta}\ \varphi^{-1}\partial_t\overline{v})\ dx\ dy\ dt\right)$$

$$0 = \Re\left(\int_\kappa^\tau \int_\Omega e^{-2s\eta}\ f\ \varphi^{-1}\partial_t\overline{v}\ dx\ dy\ dt\right) - 2s\Re\left(\int_\kappa^\tau \int_\Omega\ c\ \varphi^{-1}e^{-2s\eta}\ \partial_t\overline{v}\nabla v\cdot\nabla\eta\ dx\ dy\ dt\right)$$
$$-\lambda\Re\left(\int_\kappa^\tau \int_\Omega c\ e^{-2s\eta}\ \varphi^{-1}\partial_t\overline{v}\nabla v\cdot\nabla\beta\ dx\ dy\ dt\right) + \Re\left(\int_\kappa^\tau \int_\Omega c\ e^{-2s\eta}\ \varphi^{-1}\nabla v\cdot\nabla(\partial_t\overline{v})\ dx\ dy\ dt\right).$$

Note that

$$E(\tau) - E(\kappa) = \int_\Omega \int_\kappa^\tau\ c\ \partial_t(e^{-2s\eta}\ \varphi^{-1}|\nabla v|^2)\ dx\ dy\ dt.$$

Therefore we have

$$0 = \Re\left(\int_\kappa^\tau \int_\Omega e^{-2s\eta}\ f\ \varphi^{-1}\partial_t\overline{v}\ dx\ dy\ dt\right) - 2s\Re\left(\int_\kappa^\tau \int_\Omega\ c\ \varphi^{-1}e^{-2s\eta}\ \partial_t\overline{v}\nabla v\cdot\nabla\eta\ dx\ dy\ dt\right)$$
$$-\lambda\Re\left(\int_\kappa^\tau \int_\Omega c\ e^{-2s\eta}\ \varphi^{-1}\partial_t\overline{v}\nabla v\cdot\nabla\beta\ dx\ dy\ dt\right) + \frac{1}{2}E(\tau) - \frac{1}{2}E(\kappa)$$
$$+s\int_\kappa^\tau \int_\Omega c\ e^{-2s\eta}\ \varphi^{-1}\partial_t\eta|\nabla v|^2\ dx\ dy\ dt - \frac{1}{2}\int_\kappa^\tau ce^{-2s\eta}\partial_t(\varphi^{-1})|\nabla v|^2\ dx\ dy\ dt,$$

and the proof of Lemma 3.4 is complete. □

Theorem 3.5. *Let v be solution of (3.4) in the following class*

$$v \in C([0,T], H^1(\Omega)),\ \ \partial_\nu v \in L^2(0,T,L^2(\Gamma)).$$

We assume that Assumptions 2.1 and 2.2 are checked. Then there exists a positive constant $C = C(\Omega,\Gamma,T,R_1,R_2) > 0$ such that

$$E(0) \le C\left[s^2\lambda^2\int_{-T}^T\int_{\Gamma^+} e^{-2s\eta}\varphi\ \partial_\nu\beta\ |\partial_\nu v|^2\ d\sigma\ dt + s\lambda\int\int_Q e^{-2s\eta}|f|^2\right] \tag{3.11}$$

for s and λ sufficiently large.

Proof. We apply Lemma 3.4 with $\kappa = 0$ and $\tau = T$. Since $E(T) = 0$, we obtain

$$E(0) = 2\Re\left(\int_0^T \int_\Omega e^{-2s\eta}f\ \varphi^{-1}\partial_t\overline{v}\ dx\ dy\ dt\right) - \Re\left(\int_0^T \int_\Omega c\ e^{-2s\eta}(-4s\lambda + 2\lambda\varphi^{-1})\partial_t\overline{v}\ \nabla\beta\cdot\nabla v\ dx\ dy\ dt\right)$$
$$+2s\int_0^T \int_\Omega c\ e^{-2s\eta}\varphi^{-1}\partial_t\eta|\nabla v|^2\ dx\ dy\ dt - \int_0^T \int_\Omega ce^{-2s\eta}\partial_t(\varphi^{-1})|\nabla v|^2.$$

We give now estimates of the four integrals in the previous equality.

First integral: $B := 2\Re\left(\int_0^T \int_\Omega e^{-2s\eta}\varphi^{-1}f\ \partial_t\overline{v}\ dx\ dy\ dt\right)$.
Using (3.9), we have :

$$B = 2\Re\left(\int_0^T \int_\Omega e^{-2s\eta}\varphi^{-1}f\ \partial_t\overline{v}\ dx\ dy\ dt\right) = \Re\left(\int_0^T \int_\Omega e^{-2s\eta}\varphi^{-1}f\ \partial_t\overline{v}\ dx\ dy\ dt\right) \tag{3.12}$$
$$+\Re\left(-i\int_0^T \int_\Omega e^{-2s\eta}\varphi^{-1}f\ \nabla\cdot(c\ \nabla\overline{v})\ dx\ dy\ dt\right).$$

Recall that if we set $\psi = e^{-s\eta}v$, then $M\psi = e^{-s\eta}H(e^{s\eta}\psi) = M_1\psi + M_2\psi$ for $s > 0$ with

$$M_1\psi := i\partial_t\psi + \nabla\cdot(c\nabla\psi) + s^2c|\nabla\eta|^2\psi,$$
$$M_2\psi := is\partial_t\eta\psi + 2cs\nabla\eta\cdot\nabla\psi + s\nabla\cdot(c\nabla\eta)\psi.$$

So (3.12) becomes

$$B = 2\Re\left(\int_0^T \int_\Omega e^{-2s\eta}\varphi^{-1}f\ \partial_t\overline{v}\ dx\ dy\ dt\right) = \Re\left(\int_0^T \int_\Omega e^{-2s\eta}\varphi^{-1}f\ e^{s\eta}(s\partial_t\eta\overline{\psi} + \partial_t\overline{\psi})\ dx\ dy\ dt\right)$$

$$+\Re\left(-i\int_0^T\int_\Omega e^{-2s\eta}\varphi^{-1}f\ e^{s\eta}(s\nabla\cdot(c\nabla\eta)\overline{\psi}+c\ s^2|\nabla\eta|^2\overline{\psi}+2c\ s\ \nabla\eta\cdot\nabla\overline{\psi}+\nabla\cdot(c\nabla\overline{\psi}))\ dx\ dy\ dt\right)$$

$$=\Re\left(-i\int_0^T\int_\Omega e^{-2s\eta}\varphi^{-1}f\ e^{s\eta}(i\partial_t\overline{\psi}+\nabla\cdot(c\nabla\overline{\psi}))\ dx\ dy\ dt\right)$$

$$+\Re\left(\int_0^T\int_\Omega e^{-2s\eta}\varphi^{-1}f\ e^{s\eta}(s\partial_t\eta-i\ s\nabla\cdot(c\nabla\eta)-i\ c\ s^2|\nabla\eta|^2)\overline{\psi}\ dx\ dy\ dt\right)$$

$$+\Re\left(-i\int_0^T\int_\Omega e^{-2s\eta}\varphi^{-1}f\ e^{s\eta}2c\ s\ \nabla\eta\cdot\nabla\overline{\psi}\ dx\ dy\ dt\right)$$

Note that

$$i\partial_t\overline{\psi}+\nabla\cdot(c\nabla\overline{\psi})=M_1\overline{\psi}-s^2c|\nabla\eta|^2\overline{\psi}.$$

Then, we obtain

$$B=\Re\left(-i\int_0^T\int_\Omega e^{-2s\eta}\varphi^{-1}f\ e^{s\eta}(M_1\overline{\psi}-s^2c|\nabla\eta|^2\overline{\psi})\ dx\ dy\ dt\right)$$

$$+\Re\left(\int_0^T\int_\Omega e^{-2s\eta}\varphi^{-1}f\ e^{s\eta}(s\partial_t\eta-i\ s\nabla\cdot(c\nabla\eta)-i\ c\ s^2|\nabla\eta|^2)\overline{\psi}\ dx\ dy\ dt\right)$$

$$+\Re\left(-i\int_0^T\int_\Omega e^{-2s\eta}\varphi^{-1}f\ e^{s\eta}2c\ s\ \nabla\eta\cdot\nabla\overline{\psi}\ dx\ dy\ dt\right).$$

If we come back to the function v, the previous equality becomes :

$$B=\Re\left(-i\int_0^T\int_\Omega(fe^{-s\eta}\varphi^{-1}M_1(e^{-s\eta}\overline{v})-s^2c|\nabla\eta|^2e^{-2s\eta}\varphi^{-1}f\ \overline{v})\ dx\ dy\ dt\right)$$

$$+\Re\left(\int_0^T\int_\Omega e^{-2s\eta}\varphi^{-1}f\ (s\partial_t\eta-i\ s\nabla\cdot(c\nabla\eta)+i\ c\ s^2|\nabla\eta|^2)\overline{v}\ dx\ dy\ dt\right)$$

$$+\Re\left(-i\int_0^T\int_\Omega e^{-2s\eta}\varphi^{-1}f\ 2c\ s\ \nabla\eta\cdot\nabla\overline{v}\ dx\ dy\ dt\right).$$

Then there exists a positive constant $C=C(\Omega,\Gamma,T,R_1,R_2)$ such that:

$$|\int_0^T\int_\Omega e^{-2s\eta}\varphi^{-1}f\ \partial_t\overline{v}\ dx\ dy\ dt|\le C\left[s\lambda\int\int_Q e^{-2s\eta}|f|^2\ dx\ dy\ dt+\|M_1(e^{-s\eta}\overline{v})\|^2_{L^2(Q)}\right.$$
$$\left.+s^3\lambda^4\int\int_Q e^{-2s\eta}\ \varphi^2|v|^2\ dx\ dy\ dt+s\lambda\int\int_Q e^{-2s\eta}|\nabla v|^2\ dx\ dy\ dt\right].\tag{3.13}$$

Second integral: $D:=\Re\left(\int_0^T\int_\Omega c\ (-4s\lambda+2\lambda\varphi^{-1})\ e^{-2s\eta}(1+\varphi^{-1})\partial_t\overline{v}\ \nabla\beta\cdot\nabla v\ dx\ dy\ dt\right).$

We denote by $\rho:=-4s\lambda+2\lambda\varphi^{-1}$. Using (3.10), we have

$$2D=\Re\left(\int_0^T\int_\Omega c\ e^{-2s\eta}\rho\partial_t\overline{v}\ \nabla\beta\cdot\nabla v\ dx\ dy\ dt\right)+\Re\left(i\int_0^T\int_\Omega c\ e^{-2s\eta}\overline{f}\ \rho\nabla\beta\cdot\nabla v\ dx\ dy\ dt\right)$$

$$-\Re\left(i\int_0^T\int_\Omega c\ e^{-2s\eta}\rho\nabla\cdot(c\nabla\overline{v})\nabla\beta\cdot\nabla v\ dx\ dy\ dt\right).$$

If we introduce $\psi=e^{-s\eta}v$, we get:

$$2D=\Re\left(\int_0^T\int_\Omega ce^{-2s\eta}\rho\nabla v\cdot\nabla\beta(s\partial_t\eta e^{s\eta}\overline{\psi}+e^{s\eta}\partial_t\overline{\psi})\ dx\ dy\ dt\right)+\Re\left(i\int_0^T\int_\Omega ce^{-2s\eta}\nabla v\cdot\nabla\beta\ \overline{f}\ dx\ dy\ dt\right)$$

$$-\Re\left(i\int_0^T\int_\Omega ce^{-2s\eta}\rho\nabla v\cdot\nabla\beta[s^2c\overline{\psi}|\nabla\eta|^2e^{s\eta}+2sce^{s\eta}\nabla\overline{\psi}\cdot\nabla\eta+se^{s\eta}\overline{\psi}\nabla\cdot(c\nabla\eta)+e^{s\eta}\nabla\cdot(c\nabla\overline{\psi})]\right).$$

Therefore

$$2D = \Re\left(-i\int_0^T\int_\Omega ce^{-s\eta}\rho\nabla v\cdot\nabla\beta[M_1\overline{\psi}+M_2\overline{\psi}]\;dx\;dy\;dt\right)+\Re\left(i\int_0^T\int_\Omega ce^{-2s\eta}\rho\nabla v\cdot\nabla\beta\;\overline{f}\;dx\;dy\;dt\right).$$

Thus there exists a positive constant $C = C(\Omega,\Gamma,T,R_1,R_2)$ such that

$$|2D| \le C\left[s\lambda\int\int_Q[|M_1(e^{-s\eta}\overline{v})|^2+|M_2(e^{-s\eta}\overline{v})|^2]\;dx\;dy\;dt\right.$$
$$\left.+s\lambda\int\int_Q e^{-2s\eta}|\nabla v|^2+s\lambda\int\int_Q e^{-2s\eta}|f|^2\;dx\;dy\;dt\right]. \tag{3.14}$$

Two last integrals:
There exists a positive constant $C = C(\Omega,\Gamma,T,R_2)$ such that

$$\left|\int_0^T\int_\Omega ce^{-2s\eta}\partial_t(\varphi^{-1})|\nabla v|^2\;dx\;dy\;dt\right| \le C\int\int_Q e^{-2s\eta}|\nabla v|^2\;dx\;dy\;dt, \tag{3.15}$$

and

$$\left|s\int_0^T\int_\Omega c\partial_t\eta e^{-2s\eta}\varphi^{-1}|\nabla v|^2\;dx\;dy\;dt\right| \le C\;s\int\int_Q\varphi e^{-2s\eta}|\nabla v|^2\;dx\;dy\;dt. \tag{3.16}$$

Using now the Carleman estimate of Theorem 2.3 and Lemma 3.4, from (3.9)-(3.12), we deduce the existence of a positive constant $C = C(\Omega,\Gamma,T,R_1,R_2)$ such that:

$$E(0) \le C\left[s^2\lambda^2\int_{-T}^T\int_{\Gamma+}e^{-2s\eta}\varphi\;\partial_\nu\beta\;|\partial_\nu v|^2\;d\sigma\;dt+s\lambda\int\int_Q e^{-2s\eta}|f|^2\;dx\;dy\;dt\right],$$

and the proof is complete. □

3.2 Stability Estimate

Now following an idea developed in [16] for Lamé system in bounded domains, we give an underestimate for $\mathbb{E}(0)$. We adapt the proof of lemma 3.2 of [16] to an unbounded domain.

Assumption 3.6.
- *q_0 and all its derivatives up to order three are in $\Lambda(R_2)$*
- $|\nabla\beta\cdot\nabla q_0| \in \Lambda(R_1)$

Lemma 3.7. *We consider the first order partial differential operator*

$$(P_0g)(x,y) = \partial_x q_0(x,y)\partial_x g(x,y)+\partial_y(x,y)\partial_y g(x,y),\;P_0g := \nabla q_0\cdot\nabla g$$

where q_0 satisfies Assumptions 3.1, 3.6. Then there exist positive constants $\lambda_1 > 0$, $s_1 > 0$ and $C = C(\Omega,\Gamma,T,R_1,R_2)$ such that for all $\lambda \ge \lambda_1$ and $s \ge s_1$

$$s^2\lambda^2\int_\Omega\varphi_0\;e^{-2s\eta_0}|g|^2\;dx\;dy \le C\int_\Omega\varphi_0^{-1}\;e^{-2s\eta_0}\;|P_0g|^2\;dx\;dy$$

with $\eta_0(x,y) := \eta(x,y,0)$, $\varphi_0(x,y) := \varphi(x,y,0)$ and for $g \in H_0^1(\Omega)$.

Proof. Let $g \in H_0^1(\Omega)$. We denote by $w = e^{-s\eta_0}g$ with $\eta_0 := \eta(x,y,0)$ and $Q_0w = e^{-s\eta_0}P_0(e^{s\eta_0}w)$, so we get $Q_0w = swP_0\eta_0+P_0w$. Therefore, we have:

$$\int_\Omega\varphi_0^{-1}\;Q_0w\;\overline{Q_0w}\;dx\;dy = s^2\int_\Omega\varphi_0^{-1}\;|w|^2|P_0\eta_0|^2\;dx\;dy+\int_\Omega\varphi_0^{-1}\;|P_0w|^2\;dx\;dy$$
$$+2s\Re\left(\int_\Omega\varphi_0^{-1}\;w\;P_0\eta_0\;\overline{P_0w}\;dx\;dy\right)$$

$$= s^2\lambda^2 \int_\Omega \varphi_0 \, |w|^2(\nabla q_0 \cdot \nabla\beta)^2 \, dx \, dy + \int_\Omega \varphi_0^{-1} \, |P_0 w|^2 \, dx \, dy - s\lambda \int_\Omega \nabla q_0 \cdot \nabla\beta \, \nabla q_0 \cdot \nabla(|w|^2) \, dx \, dy.$$

So, integrating by parts, we obtain

$$\int_\Omega \varphi_0^{-1} \, Q_0 w \, \overline{Q_0 w} \, dx \, dy = s^2\lambda^2 \int_\Omega \varphi_0 \, |w|^2(\nabla q_0 \cdot \nabla\beta)^2 \, dx \, dy + \int_\Omega \varphi_0^{-1} \, |P_0 w|^2 \, dx \, dy$$

$$+ s\lambda \int_\Omega \nabla \cdot (\nabla q_0 \cdot \nabla\beta \nabla q_0)|w|^2 \, dx \, dy.$$

Thus

$$\int_\Omega \varphi_0^{-1} \, e^{-2s\eta_0}|P_0 g|^2 \, dx \, dy \geq s^2\lambda^2 \int_\Omega \varphi_0 \, |\nabla\beta\cdot\nabla q_0|^2 e^{-2s\eta_0}|g|^2 \, dx \, dy + s\lambda \int_\Omega \nabla\cdot(P_0\beta \, \nabla q_0) e^{-2s\eta_0}|g|^2 \, dx \, dy.$$

Using Assumptions 3.1 and 3.6, we can conclude for s and λ sufficiently large. $\square$

Then, we deduce the following result.

Lemma 3.8. *Let u be solution of (3.4). We assume that Assumptions 2.2, 3.1 and 3.6 are satisfied. Then there exists a positive constant $C = C(\Omega, \Gamma, T, R_1, R_2)$ such that for s and λ sufficiently large, the two following estimates hold true*

$$s^2\lambda^2 \int_\Omega \varphi_0 \, e^{-2s\eta_0}|\gamma|^2 \, dx \, dy \leq C \int_\Omega \varphi_0^{-1} \, e^{-2s\eta_0}|\partial_t u(x,y,0)|^2 \, dx \, dy, \tag{3.17}$$

$$s^2\lambda^2 \int_\Omega e^{-2s\eta_0}|\nabla\gamma|^2 \, dx \, dy \leq C \int_\Omega \varphi_0^{-1} \, e^{-2s\eta_0} \left(|\nabla(\partial_t u(x,y,0))|^2 + |\gamma|^2\right) \, dx \, dy, \tag{3.18}$$

for $\gamma \in H_0^2(\Omega)$.

Proof. We apply Lemma 3.7 to the first order partial differential equations satisfied by

- γ given by the initial condition in (3.4)

$$P_0\gamma := \partial_x q_0 \partial_x \gamma + \partial_y q_0 \partial_y \gamma = i\partial_t u(x,y,0) - \gamma\Delta q_0,$$

- $\partial_x \gamma$ given by the x-derivative of the initial condition in (3.4)

$$\begin{aligned} P_0\partial_x\gamma &:= \partial_x q_0 \partial_x(\partial_x\gamma) + \partial_y q_0 \partial_y(\partial_x\gamma) \\ &= i\partial_t(\partial_x u(x,y,0)) - \partial_x\gamma(\Delta q_0 + \partial_{xx}q_0) - \partial_y\gamma\partial_{xy}q_0 - \gamma\partial_x(\Delta q_0), \end{aligned}$$

- $\partial_y \gamma$ given by the y-derivative of the initial condition in (3.4)

$$\begin{aligned} P_0\partial_y\gamma &:= \partial_x q_0 \partial_x(\partial_y\gamma) + \partial_y q_0 \partial_y(\partial_y\gamma) \\ &= i\partial_t(\partial_y u(x,y,0)) - \partial_y\gamma(\Delta q_0 + \partial_{yy}q_0) - \partial_x\gamma\partial_{xy}q_0 - \gamma\partial_y(\Delta q_0). \end{aligned}$$

Then using Lemma 3.7 and Assumptions 3.1, 3.6, the proof of Lemma 3.8 is complete. $\square$

Theorem 3.9. *Let q and $\widetilde{q}$ be solutions of (3.1) and (3.2) such that $c - \widetilde{c} \in H_0^2(\Omega)$. We assume that Assumptions 2.1, 2.2, 3.2, 3.1 and 3.6 are satisfied. Then there exists a positive constant $C = C(\Omega, \Gamma, T, R_1, R_2)$ such that for s and λ sufficiently large,*

$$\int_\Omega \varphi_0 \, e^{-2s\eta_0}(|c - \widetilde{c}|^2 + |\nabla(c - \widetilde{c})|^2) \, dx \, dy \leq C \int_{-T}^{T} \int_{\Gamma+} \varphi \, e^{-2s\eta}\partial_\nu\beta \, |\partial_\nu(\partial_t q - \partial_t \widetilde{q})|^2 \, d\sigma \, dt.$$

Proof. Adding (3.17) and (3.18) we obtain using the estimate (3.7) for $|\mathcal{I}|$ and the energy estimate (3.11) for $E(0)$

$$s^2\lambda^2 \int_\Omega \varphi_0\ e^{-2s\eta_0} \left(|\nabla\gamma|^2 + |\gamma|^2\right)\ dx\ dy \le C \int_\Omega \varphi_0^{-1}\ e^{-2s\eta_0} \left(|\nabla(\partial_t u(x,y,0))|^2 + |\partial_t u(x,y,0)|^2\right)\ dx\ dy$$

$$\le C(|\mathcal{I}| + E(0))$$

$$\le Cs^{-3/2}\lambda^{-2} \int_\Omega e^{-2s\eta_0}(|\gamma|^2 + |\nabla\gamma|^2)\ dx\ dy + Cs^{-1/2}\lambda^{-1} \int_{-T}^{T}\int_{\Gamma+} e^{-2s\eta}\varphi\ \partial_\nu\beta\ |\partial_\nu v|^2\ d\sigma\ dt$$

$$+Cs^2\lambda^2 \int_{-T}^{T}\int_{\Gamma+} e^{-2s\eta}\varphi\ \partial_\nu\beta\ |\partial_\nu v|^2\ d\sigma\ dt + Cs\lambda \int\int_Q e^{-2s\eta}|f|^2.$$

So we get

$$s^2\lambda^2 \int_\Omega \varphi_0\ e^{-2s\eta_0} \left(|\nabla\gamma|^2 + |\gamma|^2\right)\ dx\ dy \le Cs^2\lambda^2 \int_{-T}^{T}\int_{\Gamma+} e^{-2s\eta}\ \varphi\ \partial_\nu\beta\ |\partial_\nu v|^2\ d\sigma\ dt$$

$$+Cs\lambda \int\int_Q e^{-2s\eta}\ |\nabla\cdot(\gamma\nabla\partial_t\widetilde{q})|^2\ dx\ dy\ dt$$

$$\le Cs^2\lambda^2 \int_{-T}^{T}\int_{\Gamma+} e^{-2s\eta}\ \varphi\ \partial_\nu\beta\ |\partial_\nu v|^2\ d\sigma\ dt + Cs\lambda \int\int_Q e^{-2s\eta}\ \left(|\nabla\gamma|^2 + |\gamma|^2\right)\ dx\ dy\ dt.$$

Then, for s and λ sufficiently large, the theorem is proved. □

Remark 3.10. *This result is also available for the heat equation in bounded or unbounded domains. Note that all the previous results proved in* $\mathbb{R}\times(-\frac{d}{2},\frac{d}{2})$ *are available in* $\mathbb{R}^n\times(-\frac{d}{2},\frac{d}{2})$ *for* $n\ge 2$ *if we adapt the regularity properties of the initial and boundary conditions.*

References

[1] A.L. Bukhgeim, Volterra Equations and Inverse Problems, Inverse and Ill-Posed Problems Series, VSP, Utrecht, 1999.

[2] L. Baudouin and J.P. Puel, *Uniqueness and stability in an inverse problem for the Schrödinger equation*, Inverse Problems, 18, 1537–1554 (2002).

[3] A.L. Bukhgeim and M.V. Klibanov, Uniqueness in the large of a class of multidimensional inverse problems, Soviet Math. Dokl., 17, 244-247, 1981.

[4] H. Brezis, *Analyse Fonctionnelle. Théorie et Applications*, Masson (1983).

[5] J.L. Buchanan, R.P. Gilbert, A. Wirgin and Y.S. Xu, *Marine acoustics. Direct and inverse problems.* Society for Industrial and Applied Mathematics, Philadelphia, PA (2004).

[6] M. Burgos-Gonzalez and L. de Tereza, *Some results on null controllability for the heat equation in unbounded domains*, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

[7] B. Chenaud, P. Duclos, P. Freitas and D. Krejcirik, *Geometrically Induced Discrete Spectrum in Curved Tubes*, Differential Geom. Appl. 23, N.2, 95–105 (2005).

[8] P. Cristini and A. Wirgin, *Identification of the size, proportions and location of a soft body of revolution in a shallow-water waveguide*, Inverse Problems, 16, 1727–1739 (2000).

[9] M. Cristofol and P. Gaitan, *Inverse Problem for a Perturbed Stratified Strip in two dimension*, Math. Meth. Appl. Sci 27, 1–17 (2004).

[10] M. Cristofol, P. Gaitan and V. Iftimie, *Inverse Problem for the Schrödinger Operator in a Layer*, Revue Roumaine de Mathématiques Pures et Appliquées t.L, 153–180 (2005).

[11] P. Duclos, P. Exner and D. Krejcirik, *Bound States in Curved Quantum Layers*, Commun. Math. Phys. 223, 13–28 (2001).

[12] P. Exner and P. Seba, *Bound states in curved quantum waveguides* J. Math. Phys., 30 (11), 2574–2580 (1989).

[13] M. Ikehata, *Inverse Conductivity Problem in the Infinite Slab*, Inverse Problems, 17, 437-454 (2001).

[14] V. Isakov, *Inverse problems for partial differential equations*, Springer-Verlag, 1998.

[15] M. Ikehata, G.N. Makrakis and G. Nakamura, *Inverse Boundary Problems for Ocean Accoustics*, Math. Meth. Appli. Sci. 24, 1–8 (2004).

[16] O. Yu. Immanuvilov, V. Isakov and M. Yamamoto, *An Inverse Problem for the Dynamical Lamé system with two set of boundary data*, CPAM, Vol LVI, 1366–1382 (2003).

[17] O. Yu. Immanuvilov and M. Yamamoto, *Carleman estimates for the non-stationary Lamé system and the application to an inverse problem*, ESAIM Control Optim. Calc. Var. 11, no. 1, 1–56 (2005).

[18] C. Kenig, J. Sjöstrand and G. Uhlmann, *The Calderón problem with partial data*, to appear in Ann. of Math.

[19] M.V. Klibanov, *Inverse problems in the large and Carleman bounds*, Differential Equations, 20, 755-760, 1984.

[20] M.V. Klibanov, *Inverse problems and Carleman estimates, Inverse Problems*, 8, 575- 596, 1992.

[21] M.V. Klibanov and M. Yamamoto, *Lipshitz stability for an inverse problem for an acoustic equation*, Applicable Analysis, 85, 515–538 (2006).

[22] M.V. Klibanov and A. Timonov, *Carleman estimates for coefficient inverse problems and numerical applications*, Inverse and Ill-posed series,VSP, Utrecht, 2004.

[23] I. Lasiecka, R. Triggiani and X. Zhang, *Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates* , J. Inv. Ill-Posed Problems, Vol 11 (3), 1–96 (2003).

[24] J.-P. Puel and M. Yamamoto, *On a global estimate in a linear inverse hyperbolic problem, Inverse Problems*, 12, 995-1002, 1996.

[25] A. Wirgin, *Inverse problems in underwater acoustics*, Inverse Problems, 16, 1619–1625 (2000).